JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **165**, 62–70 (1992)

Geometry and Integration for Operator Valued Measures

Mª E. BALLVÉ

Departamento de Matematicas Fundamentales, U.N.E.D., Madrid, Spain

Submitted by Dorothy Maharam Stone

Received April 12, 1989

The relationship between the average range of the measure defined by an m-integrable function, the core, and the essential range of the function, are studied where m is an operator valued measure. A measurability criterion, a mean value theorem for the Dobrakov integral, and a Radon–Nikodym type theorem are also proved. © 1992 Academic Press, Inc.

The core and the essential range of a function are very useful in the study of different types of measurability in the setting of scalar measures, along the lines of Pettis's theorem [10] (see, for instance, [6, 8]). The main object of this paper is to study the analogue of these sets for operator valued measures. For this, the measure is assumed to satisfy the Price's axiom (see [11]) (nevertheless, Proposition 8 and Theorem 11 also hold without this assumption), which permits one to define the average range of a measure with values in a Banach space E with respect to a measure m with values in the space $L(E, E)$, of the continuous linear functions from E to E, and the convex hull of a subset of E with respect to the measure m. The core (with respect to m) of an E-valued function is defined from the last concept, and several relations are stated between the average range of a measure defined by an m-integrable E-valued function f, the core of f, and its essential range, and also between their closed m-convex hulls. As an application, a mean value theorem and a criterion for measurability are stated.

The extension, introduced here, of the concept of the average range of a vector function, from scalar to vector measures, has enabled us to obtain a Radon–Nikodym type theorem reminiscent of the classical Radon–Nikodym theorem for the Bochner integral, while the known theorems of this type are obtained by quite different methods (see [1, 2, 7, 9]).

The integral used in this paper coincides essentially with that of Dobrakov [5], but it has been presented in such a way as to indicate that

0022-247X/92 $3.00

it is a special case of an integral for locally convex spaces (see, for instance, [3, 6, 7]), which permits the extension of several results of this paper to Fréchet spaces and some of them to quasi-complete locally convex spaces.

Let us denote by Σ a σ-algebra of subsets of a set Ω and by E and F two Banach spaces. $L(E, F)$ will denote the Banach space of all bounded linear operators from E to F, endowed with the strong operator topology. We consider the evaluation mapping from $ExL(E, F)$ into F and a countably additive measure $m: \Sigma \to L(E, F)$ of bounded semivariation (i.e., $\|m\|\ (\Omega) < +\infty$) and continuous (i.e., $\lim_{E_n \downarrow \varnothing} \|m\|\ (E_n) = 0$). Recall that the semivariation of the measure m is defined by

$$\|m\|\ (A) = \sup \left\| \sum_{i=1}^{n} m(A_i)\, x_i \right\|,$$

where the supremum is taken over all the finite and measurable partitions of $A \in \Sigma$ and all the finite families $\{x_i\}_{i=1}^{n} \subseteq E$ with $\|x_i\| \leqslant 1$ for $i = 1, ..., n$.

A set $N \in \Sigma$ is said to be m-null iff $\|m\|\ (N) = 0$. From now on, $\mathfrak{N}_m$ will denote the family of all m-null subsets of Ω, $\Sigma_A = \{B \in \Sigma : B \subseteq A\}$, $\Sigma'_A = \{B \in \Sigma_A : m(B) \neq 0\}$, $\Sigma_A^+ = \Sigma_A - \mathfrak{N}_m$ for every $A \in \Sigma$, $\Sigma' = \Sigma'_\Omega$, and $\Sigma^+ = \Sigma_\Omega^+$. Let us remark that $N \in \mathfrak{N}_m$ iff $m(B) = 0$ for every $B \in \Sigma_N$.

Definition 1. A function $f: \Omega \to E$ is said to be *u-simple* ($f \in S_u$) if it is a uniform limit of a sequence of simple functions (the simple functions, their integrals, and the integral of the u-simple functions are defined in the standard way), and we say that a function $f: \Omega \to E$ is m-measurable if for every $\varepsilon > 0$ there exists $A \in \Sigma$ such that $\|m\|\ (\Omega - A) \leqslant \varepsilon$ and the function $f\chi_A$ is u-simple.

A function $f: \Omega \to E$ is said to be m-integrable if it is m-measurable and

$$\lim_{\substack{\|m\|\ (A) \to 0 \\ A \in \Sigma}} \|m\|\ (f, A) = 0,$$

where

$$\|m\|\ (f, A) = \sup \left\{ \left\| \int_A g\, dm \right\| : g \in S_u \mid \|g(t)\| \leqslant \|f(t)\|\ \forall t \in A \right\}.$$

If f is an m-integrable function, then

$$\int_A f\, dm = \lim_{K \in \{B \in \Sigma : f\chi_B \in S_u\}} \int_A f\chi_K\, dm$$

for every $A \in \Sigma$ and $m_f: \Sigma \to F$ will be as usual, the measure defined by

$$m_f(A) = \int_A f\, dm \qquad (A \in \Sigma).$$

From now, let us assume that $E=F$, $\Omega\in\Sigma^+$, and that the measure m satisfies the Price's axiom, i.e., for every $A\in\Sigma$, $m(A)$ is the null function or it is an homeomorphism.

DEFINITION 2. If $\alpha\colon\Sigma\to E$ is a vector measure, we define its *average range* (with respect to m) as

$$\mathscr{A}_\alpha(A)=\{m(B)^{-1}[\alpha(B)]\colon B\in\Sigma_A\}$$

for every $A\in\Sigma^+$. We say that the measure α has *locally small average range* if for every $\varepsilon>0$ and $A\in\Sigma^+$ there exists $B\in\Sigma_A^+$ such that the diameter of $\mathscr{A}_\alpha(B)$ is less or equal than ε.

The *m-convex hull* of a subset $C\subseteq E$ will be the set $cm(C)$ of all vectors $y\in E$ such that

$$y=m(B)^{-1}\left[\sum_{i=1}^{n} m(B_i)\,x_i\right]$$

for some finite and disjoint family $\{B_i\}_{i=1}^n\subset\Sigma$ with $B=\bigcup_{i=1}^n B_i\in\Sigma'$, and some finite subset $\{x_i,...,x_n\}\subseteq C$. As usual, $\overline{cm}(C)$ will denote the closure of $cm(C)$. Clearly, if $C\subseteq D\subseteq E$ then $C\subseteq cm(C)\subseteq cm(D)$.

If $f\colon\Omega\to E$ is a function, the *core* of f on $A\in\Sigma$ will be the set

$$\operatorname{cor}_f(A)=\bigcap\{\overline{cm}[f(A-N)]\colon N\in\mathfrak{N}_m\}.$$

It follows immediately from the definition that $\operatorname{cor}_f(A)\subseteq\overline{cm}[f(A)]$, $\operatorname{cor}_f(B)\subseteq\operatorname{cor}_f(A)$, and $\operatorname{cor}_f(A)=\operatorname{cor}_f(A-N)$ for every $A\in\Sigma$, $B\in\Sigma_A$, and $N\in\mathfrak{N}_m$. Therefore, $\operatorname{cor}_f(A)=\operatorname{cor}_g(A)$ if $g\colon\Omega\to E$ is another function such that $f=g$ a.e. We say that a function f has *locally small core* if for every $A\in\Sigma^+$ and $\varepsilon>0$ there exists $B\in\Sigma_A^+$ such that the diameter of $\operatorname{cor}_f(B)$ is less or equal than ε.

LEMMA 3. *If $A\in\Sigma$ and $f\in S_u$, then there exists a sequence (f_n) of simple functions which is uniformly convergent to f on A and verifies that $\bigcup_{n\in\mathbb{N}} f_n(A)\subseteq f(A)$.*

THEOREM 4. *Let be $A\in\Sigma^+$ and assume that one of the following conditions hold:*

4.1. *$f\chi_A$ is u-simple function.*

4.2. *$f\chi_A$ is an m-integrable function such that $\mathscr{A}_{m_f}(A)$ is a bounded set of E.*

Then

$$\mathscr{A}_{m_f}(A) \subseteq \operatorname{cor}_f[f(A)]. \tag{4.1}$$

Proof. Suppose first that the condition 4.1 holds, then it follows from Lemma 3 that there exists a sequence of simple functions

$$f_n = \sum_{i=1}^{k_n} x_i^n \chi_{A_i^n}$$

such that it converges uniformly to f on A, and $\bigcup_{n \in \mathbb{N}} f_n(A) \subseteq f(A)$. Therefore,

$$\begin{aligned} m(B)^{-1}\left(\int_B f\,dm\right) &= m(B)^{-1}\left(\lim_n \int_B f_n\,dm\right) \\ &= \lim_n m(B)^{-1}\left[\sum_{i=1}^{k_n} m(A_i^n \cap B)\, x_i^n\right] \\ &\in \overline{cm}[f(A)] \end{aligned}$$

for every $B \in \Sigma'_A$ and (4.1) holds.

If condition 4.2 is verified and $B \in \Sigma'_A$, then there exists an increasing sequence $(B_n)_{n \in \mathbb{N}} \subset \Sigma'_B$ such that $\|m\|\,(B - \bigcup_{n \in \mathbb{N}} B_n) = 0, f\chi_{B_n} \in S_u$, and

$$x = m(B)^{-1}\left(\int_B f\,dm\right) = \lim_n m(B)^{-1}\left(\int_{B_n} f\,dm\right).$$

Then, it follows from 4.1 that

$$y_n = m(B_n)^{-1}\left(\int_{B_n} f\,dm\right) \in \overline{cm}[f(A)]$$

and, since $\lim_n m(B)^{-1}\, m(B_n)$ is the identity on E, it follows that for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\begin{aligned} \|x - y_n\| &\leqslant \|x - m(B)^{-1}\, m(B_n)(y_n)\| + \|m(B)^{-1}\, m(B_n)(y_n) - y_n\| \\ &\leqslant \varepsilon(1 + \|y_n\|) \end{aligned}$$

for every $n \geqslant n_0$, from where it follows immediately that $x \in \overline{cm}[f(A)]$, since $(y_n)_{n \in \mathbb{N}} \subseteq \mathscr{A}_{m_f}(A)$.

The result now follows immediately, since

$$\mathscr{A}_{m_f}(A - N) \subseteq \overline{cm}[f(A - N)],$$

and $\mathscr{A}_{m_f}(A - N) = \mathscr{A}_{m_f}(A)$, because $m(B)^{-1} = m(B - N)^{-1}$ for every $B \in \Sigma'_A$.

PROPOSITION 5. *If f is an m-integrable function then*

$$\mathscr{A}_{m_f}(A) \subseteq \overline{cm}[f(A) \cup \{0\}]$$

for every $A \in \Sigma^+$.

Proof. It is enough to proceed as in the proof of the Theorem 4 and to note that for every simple function $g = \sum_{i=1}^{n} x_i \chi_{A_i}$ with $g(A) \subseteq f(A)$ we have that

$$\begin{aligned} m(B)^{-1}\left(\int_{B \cap A} g\,dm\right) &= m(B)^{-1}\left[\sum_{i=1}^{n} m(A_i \cap B \cap A)\, x_i\right] \\ &= m(B)^{-1}\left[\sum_{i=1}^{n} m(A_i \cap B \cap A)\, x_i + m(B - A)\, 0\right] \\ &\in \overline{cm}[f(A) \cup \{0\}]. \end{aligned}$$

THEOREM 6. *If f is an m-integrable function then for every $A \in \Sigma$ there exists $x \in \overline{cm}[f(A) \cup \{0\}]$ such that*

$$\int_A f\,dm = m(A)\, x. \tag{6.1}$$

Moreover, if $A \in \Sigma^+$ then the vector x is unique and if $\mathscr{A}_{m_f}(A)$ is a bounded subset of E, then $x \in \overline{cm}[f(A)]$.

Proof. It follows immediately from Theorem 4 and Proposition 5.

Remark. Note that if f is an m-integrale function and $N \in \mathfrak{N}_m - \{\varnothing\}$ then the function $g = f\chi_{\Omega - N}$ is also m-integrale, $m_g = m_f$, $\mathscr{A}_{m_g}(A) = \mathscr{A}_{m_f}(A)$ for every $A \in \Sigma^+$ and $0 \in g(\Omega)$.

DEFINITION 7. Let $f\colon \Omega \to E$ a function, the *essential range* of f on $A \in \Sigma$ will be the set

$$\mathrm{er}_f(A) = \bigcap \{\overline{f(A - N)} : N \in \mathfrak{N}_m\}.$$

It is clear that $\mathrm{er}_f(B) \subseteq \mathrm{er}_f(A) \subseteq \overline{f(A)}$ for every $B \in \Sigma_A$ and $\mathrm{er}_f(A) = \mathrm{er}_g(A)$ for every function $g\colon \Omega \to E$ such that $f = g$ a.e. on A. It follows immediately from the definition that

$$\mathrm{er}_f(A) \subseteq \mathrm{cor}_f(A). \tag{7.1}$$

The function f is said to be regular on $A \in \Sigma^+$ if $\mathrm{er}_f(A) \neq \varnothing$ for every $B \in \Sigma_A^+$.

PROPOSITION 8. *If* $A \in \Sigma$ *and* $f: \Omega \to E$ *is a function such that* $f\chi_A$ *is m-measurable, then the following assertions hold*:

8.1. $\mathrm{er}_f(A) = \{x \in E: \{t \in A: \|f(t) - x\| \leqslant \varepsilon\} \notin \mathfrak{N}_m$, *for every* $\varepsilon > 0\}$.

8.2. *If the function* $f\chi_A$ *is u-simple then* $\mathrm{er}_f(A)$ *is compact.*

8.3. *For every* $\varepsilon > 0$ *there exists* $K \in \Sigma_A$ *such that* $\|m\|\,(A - K) \leqslant \varepsilon$ *and* $\mathrm{er}_f(K)$ *is compact, and therefore* $\mathrm{er}_f(A)$ *is Lindelöf.*

Proof. 8.1. Let be $x \in \mathrm{er}_f(A)$, then if there exists $\varepsilon > 0$ such that $N_{x,\varepsilon} = \{t \in A: \|f(t) - x\| \leqslant \varepsilon\} \in \mathfrak{N}_m$ then $x \notin \overline{f(A - N_{x,\varepsilon})}$ which is a contradiction. Moreover, if $x \notin \mathrm{er}_f(A)$ then there exists $N_x \in \mathfrak{N}_m$ such that $x \notin \overline{f(A - N_x)}$ and therefore, there is $\varepsilon > 0$ for which $\{t \in A: \|f(t) - x\| \leqslant \varepsilon\} \subseteq N_x$ holds, which is a contradiction.

8.2. For every $\varepsilon > 0$ there exists a simple function $g = \sum_{i=1}^{n} x_i \chi_{A_i}$ such that $\|f(t) - g(t)\| \leqslant \varepsilon$ for every $t \in A$ and $g(A) \subseteq f(A)$. Therefore, $\overline{f(A)} \subseteq \bigcup_{i=1}^{n} [x_i + B(0, \varepsilon)]$, where $B(0, \varepsilon) = \{x \in E: \|x\| \leqslant \varepsilon\}$, from which it follows that $\mathrm{er}_f(A)$ is compact, since it is a closed subset of $\overline{f(A)}$.

8.3. Follows immediately from 8.2.

THEOREM 9. *If* $A \in \Sigma^+$ *and* f *is a regular function on* A *such that* $f\chi_A$ *is m-measurable, then*

$$\mathrm{cor}_f(A) = \overline{cm}[\mathrm{er}_f(A)]. \tag{9.1}$$

Proof. Since the function f is regular on A and the measure m is continuous then it follows from 8.1 and the Zorn's axiom that for every $n \in \mathbb{N}$ there exists a disjoint sequence $(A_k^n)_{k \in \mathbb{N}} \subset \Sigma_A^+$ and a sequence $(y_k^n)_{k \in \mathbb{N}} \subseteq \mathrm{er}_f(A)$ such that $N_n = A - \bigcup_{k \in \mathbb{N}} A_k^n \in \mathfrak{N}_m$ and $f(A_k^n) \subseteq y_k + B(0, 1/n)$ for every $k \in \mathbb{N}$. Then if $x \in \mathrm{cor}_f(A) \subseteq \overline{cm}[f(A - N)]$, with $N = \bigcup_{n \in \mathbb{N}} N_n \in \mathfrak{N}_m$, for every $\varepsilon > 0$ there exists $\{t_1, ..., t_r\} \subseteq A - N$ and a disjoint family $\{B_1, ..., B_r\} \subset \Sigma$ such that $B = \bigcup_{i=1}^{r} B_i \in \Sigma'$ and

$$\left\| x - m(B)^{-1} \left[\sum_{i=1}^{r} m(B_i)\, f(t_i) \right] \right\| \leqslant \varepsilon.$$

Then, there exists $n_0, k_1, ..., k_r \in \mathbb{N}$ such that $\|m(B)^{-1}\|\ \|m\|\,(\Omega) \leqslant \varepsilon n/2$, $t_i \in A_{k_i}^{n_0}$ for $i = 1, ..., r$, and

$$y = m(B)^{-1} \left[\sum_{i=1}^{r} m(B_i)\, y_{k_i}^{n_0} \right] \in cm[\mathrm{er}_f(A)]$$

satisfies

$$\begin{aligned}\|x-y\| &\leqslant \left\| x - m(B)^{-1}\left[\sum_{i=1}^{r} m(B_i)\, f(t_i)\right]\right\| \\ &\quad + \left\| m(B)^{-1}\left[\sum_{i=1}^{r} m(B_i)(f(t_i)-y_{k_i}^{n_0})\right]\right\| \\ &\leqslant \varepsilon/2 + (\|m(B)^{-1}\|\ \|m\|\ (\Omega))/n \\ &\leqslant \varepsilon.\end{aligned}$$

Therefore, $x \in \overline{cm}[\mathrm{er}_f(A)]$ and $\mathrm{cor}_f(A) \subseteq \overline{cm}[\mathrm{er}_f(A)]$.

Moreover, if $x \in \overline{cm}[\mathrm{er}_f(A)]$ then for every $\varepsilon > 0$ there exists a disjoint family $\{B_1, \ldots, B_n\} \subset \Sigma$ and $\{y_1, \ldots, y_n\} \in \mathrm{er}_f(A)$ such that $B = \bigcup_{i=1}^{n} B_i \in \Sigma'$ and

$$\left\| x - m(B)^{-1}\left[\sum_{i=1}^{n} m(B_i)\, y_i\right]\right\| \leqslant \varepsilon/2.$$

Let $n_0 \in \mathbb{N}$ be such that $\|m(B)^{-1}\|\ \|m\|\ (\Omega) \leqslant \varepsilon n_0/2$, then since $y_i \in \mathrm{er}_f(A)$ for $i = 1, \ldots, n$, the set $A_i = \{t \in A \colon \|f(t) - y_i\| \leqslant 1/n_0\} \notin \mathfrak{N}_m$ and if $N \in \mathfrak{N}_m$ there exists $t_i \in A_i - N$ such that

$$y = m(B)^{-1}\left[\sum_{i=1}^{n} m(B_i)\, f(t_i)\right] \in \overline{cm}[f(A-N)]$$

and

$$\begin{aligned}\|x-y\| &\leqslant \varepsilon/2 + \|m(B)^{-1}\| \left\| \sum_{i=1}^{n} m(B_i)(y_i - f(t_i))\right\| \\ &\leqslant \varepsilon/2 + (\|m(B)^{-1}\|\ \|m\|\ (\Omega))/n_0 \\ &\leqslant \varepsilon,\end{aligned}$$

from which it follows that $x \in \overline{cm}[f(A-N)]$ for every $N \in \mathfrak{N}_m$ and, therefore, $x \in \mathrm{cor}_f(A)$, $\overline{cm}[\mathrm{er}_f(A)] \subseteq \mathrm{cor}_f(A)$ and (9.1) holds.

COROLLARY 10. *If $A \in \Sigma^+$, f is a regular function on A such that $f\chi_A$ is m-integrable and $\mathscr{A}_{m_f}(A)$ is bounded, then*

$$\mathscr{A}_{m_f}(A) \subseteq \overline{cm}[\mathrm{er}_f(A)]. \tag{10.1}$$

Proof. It is an immediate consequence of the Theorems 4 and 9.

THEOREM 11. *If $f\colon \Omega \to E$ is a regular function on Ω and $\|f-x\|$ is measurable (i.e., $\|f-x\|^{-1}(B) \in \Sigma$ for every Borel subset of $\mathbb{R}$) for every $x \in E$, then the function f is m-measurable.*

Proof. Under these conditions 8.1 holds and, proceeding as in the first part of the proof of Theorem 9, for every $n \in \mathbb{N}$, we can find a disjoint sequence $(A_k^n)_{k\in\mathbb{N}} \subset \Sigma$ and a sequence $(y_k^n)_{k\in\mathbb{N}} \subseteq E$ such that $N_n = \Omega - \bigcup_{n\in\mathbb{N}} A_k^n \in \mathfrak{N}_m$ and $f(A_k^n) \subseteq y_k^n + B(0, 1/n)$. Then the function

$$g_n = \sum_{k\in\mathbb{N}} y_k^n \chi_{A_k^n}$$

is m-measurable and the sequence $(g_n)_{n\in\mathbb{N}}$ is uniformly convergent to f on $A - \bigcup_{n\in\mathbb{N}} N_n$, and therefore the function f is m-measurable.

THEOREM 12. *Let $\alpha\colon \Sigma \to E$ be a countably additive measure having a locally small average range (with respect to m). If $\alpha \ll m$ (i.e., $B \in \mathfrak{N}_m$ implies $\alpha(B) = 0$) then there exists an m-integrable function $f\colon \Omega \to E$ such that $\alpha = m_f$ (i.e., the measure α has a Radon–Nikodym derivative with respect to m).*

Proof. Proceeding in standard way, for every $n \in \mathbb{N}$ we can find a disjoint sequence $(A_k^n)_{k\in\mathbb{N}} \subseteq \Sigma^+$ such that $\|m\|\,(\Omega - \bigcup_{k\in\mathbb{N}} A_k^n) = 0$, the diameter of $\mathscr{A}_\alpha(A_k^n)$ is less or equal than $1/n$ and for every $n, k \in \mathbb{N}$ there exists $j_k \in \mathbb{N}$ which verifies that $A_k^{n+1} \subseteq A_{j_k}^n$. Let be $N_0 = \bigcup_{n\in\mathbb{N}} (\Omega - \bigcup_{k\in\mathbb{N}} A_k^n)$, then $\|m\|\,(N_0) = 0$ and it is easily verified that the functions

$$f_n = \sum_{k\in\mathbb{N}} m(A_k^n)^{-1}\,[\alpha(A_k^n)]\,\chi_{A_k^n}$$

are measurable for every $n \in \mathbb{N}$ and that the sequence $(f_n)_{n\in\mathbb{N}}$ is uniformly convergent on $\Omega - N_0$ to an m-integrable function f such that $f \cdot \chi_{N_0} \equiv 0$ (note that f_n is a bounded function for every $n \in \mathbb{N}$, and, therefore, the function f is essentially bounded).

Moreover, for every $B \in \Sigma$ and $n \in \mathbb{N}$ we have that

$$\begin{aligned}
\left\|\alpha(B) - \int_B f_n\,dm\right\| &= \Bigg\| \sum_{\substack{k=1\\ m(B\cap A_k^n)=0}}^{\infty} \alpha(B\cap A_k^n) \\
&\qquad - \sum_{\substack{k=1\\ m(B\cap A_k^n)=0}}^{\infty} m(B\cap A_k^n)[m(A_k^n)^{-1}(\alpha(A_k^n))]\Bigg\| \\
= \Bigg\| \sum_{\substack{k=1\\ m(B\cap A_k^n)=0}}^{\infty} & m(B\cap A_k^n)\,[m(A_k^n\cap B)^{-1}\,\alpha(B\cap A_k^n) - m(A_k^n)^{-1}\,\alpha(A_k^n)]\Bigg\| \\
&\leqslant (\|m\|\,(\Omega))/n,
\end{aligned}$$

from which it follows immediately that $\alpha = m_f$.

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